A Hyperbola in complex plane ?

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Question 1

A variable complex number z satisfies the equation

$||z + 2i| - |z - 2i|| = \mathbf{4}$.

Find the equation of the locus of z on the Argand diagram.

Is it a hyperbola ?

Solution

Let $z = x + iy$.

$||z + 2i| - |z - 2i|| = 4 \quad \Rightarrow \quad ||x + iy + 2i| - |x + iy - 2i|| = 4$

$\Rightarrow \quad \sqrt{x^2 + (y+2)^2} - \sqrt{x^2 + (y-2)^2} = \pm 4 \quad \Rightarrow \quad \sqrt{x^2 + (y+2)^2} = \sqrt{x^2 + (y-2)^2} \pm 4$

$\Rightarrow \quad x^2 + y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 \pm 8\sqrt{x^2 + (y-2)^2} + 16$

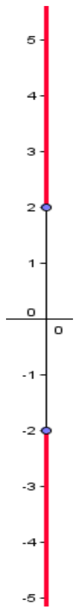
$\Rightarrow \quad y - 2 = \pm\sqrt{x^2 + (y-2)^2} \quad \Rightarrow \quad y^2 - 4y + 4 = x^2 + y^2 - 4y + 4$

$\Rightarrow \quad x^2 = 0 \quad \Rightarrow \quad x = 0$

$\therefore$ The required equation of locus of z is $x = 0$.

If you study more closely, the locus is composed of two half-lines as in the diagram on the right hand side.

It is not a hyperbola. (Or at most, a degenerated hyperbola)



If you change the question a bit, you can get a hyperboa as in the followings:

Question 2

A variable complex number z satisfies the equation $||z + 2i| - |z - 2i|| = \mathbf{2}$.

Find the equation of the locus of z on the Argand diagram.

Solution

Let $z = x + iy$.

$||z + 2i| - |z - 2i|| = 2 \quad \Rightarrow \quad ||x + iy + 2i| - |x + iy - 2i|| = 2$

$\Rightarrow \quad \sqrt{x^2 + (y+2)^2} - \sqrt{x^2 + (y-2)^2} = \pm 2 \quad \Rightarrow \quad \sqrt{x^2 + (y+2)^2} = \sqrt{x^2 + (y-2)^2} \pm 2$

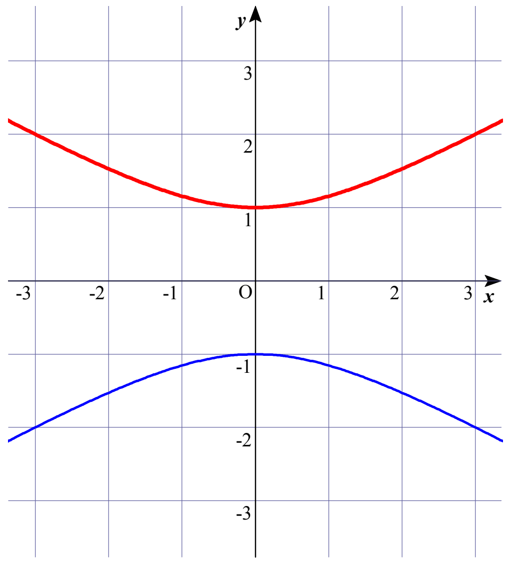
$\Rightarrow \quad x^2 + y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 \pm 4\sqrt{x^2 + (y-2)^2} + 4$

$\Rightarrow \quad 2y - 1 = \pm\sqrt{x^2 + (y-2)^2} \quad \Rightarrow \quad 4y^2 - 4y + 1 = x^2 + y^2 - 4y + 4$

$\Rightarrow \quad 3y^2 - x^2 = 3$

$\therefore$ The required equation of locus of z is $\frac{y^2}{1} - \frac{x^2}{3} = 1$, which is a **hyperbola**.

The Argand diagram is as follows:



The foci of the hyperbola are: $\pm 2i$.

Challenge

A variable complex number z satisfies the equation

$| |z + 2i| - |z - 2i| | = 8$.

Find the equation of the locus of z on the Argand diagram.

Is it a hyperbola ?

What is wrong with the following "Solution" ?

Let $z = x + iy$.

$| |z + 2i| - |z - 2i| | = 8 \quad \Rightarrow \quad | |x + iy + 2i| - |x + iy - 2i| | = 8$

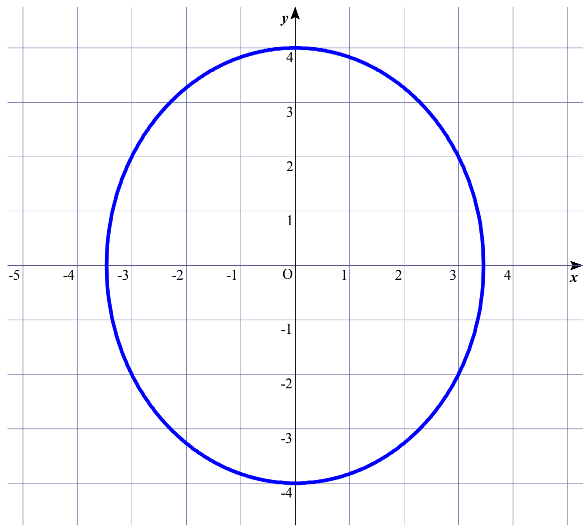
$\Rightarrow \quad \sqrt{x^2 + (y+2)^2} - \sqrt{x^2 + (y-2)^2} = \pm 8 \quad \Rightarrow \quad \sqrt{x^2 + (y+2)^2} = \sqrt{x^2 + (y-2)^2} \pm 8$

$\Rightarrow \quad x^2 + y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 \pm 16\sqrt{x^2 + (y-2)^2} + 64$

$\Rightarrow \quad 8y - 64 = \pm 16\sqrt{x^2 + (y-2)^2} \quad \Rightarrow \quad y - 8 = \pm 2\sqrt{x^2 + (y-2)^2}$

$\Rightarrow \quad y^2 - 16y + 64 = 4(x^2 + y^2 - 4y + 4) \quad \Rightarrow \quad 4x^2 + 3y^2 = 48$

$\Rightarrow \quad \frac{x^2}{12} + \frac{y^2}{16} = 1$



Is the locus an **ellipse** with foci: $\pm 2i$?

But obviously, the point 4i, which is on the locus, does not satisfy the equation:

$| |z + 2i| - |z - 2i| | = 8$

If the locus is not $\frac{x^2}{12} + \frac{y^2}{16} = 1$, what is it?

Where is the wrong step in the above calculations?

Solution of Challenge

The equation : $\left||z+2i|-|z-2i|\right| = 8$ has no locus in the complex plane .

Let $z_A = 2i$, $z_B = -2i$ and A, B be their corresponding point in Argand diagram.
Let P be any point in the complex plane .
The distance $AB = |z_A - z_B| = |2i-(-2i)| = 4$ …. (1)

By the **Triangular inequality**,
(a) If $PA > PB$, then $AB + PB > PA$. $\therefore AB > PA - PB$
(b) If $PB > PA$, then $AB + PA > PB$. $\therefore AB > PB - PA$
In any of the above case, we have $AB > |PB - PA|$ …. (2)

By (1) and (2), we get $4 > |PB - PA| = \left||z+2i|-|z-2i|\right|$

$\therefore$ $\left||z+2i|-|z-2i|\right| = 8$
has no locus in the complex plane , since the L.H.S. of the equation $< 4 \quad \forall z \in \mathbb{C}$.

The mistake of the calculations to get an ellipse comes from squaring.
You can see that the point (0, 4) which is on the ellipse $\frac{x^2}{12}+\frac{y^2}{16}=1$ does not satisfy :

$$\sqrt{x^2+(y+2)^2} = \sqrt{x^2+(y-2)^2} \pm 8 \qquad \text{(L.H.S. = 6, R.H.S. = 10 or −6)}$$

However, (0, 4) satisfies the steps that follow in the above calculations after squaring.

If we begin with $|z+2i|+|z-2i| = 8$, the calculations are as follows:
$\left||x+iy+2i|+|x+iy-2i|\right| = 8$

$$\Rightarrow \sqrt{x^2+(y+2)^2}+\sqrt{x^2+(y-2)^2} = 8 \qquad \Rightarrow \sqrt{x^2+(y+2)^2} = 8-\sqrt{x^2+(y-2)^2}$$

$$\Rightarrow x^2+y^2+4y+4 = 64-16\sqrt{x^2+(y-2)^2}+x^2+y^2-4y+4$$

$$\Rightarrow 8y-64 = -16\sqrt{x^2+(y-2)^2} \qquad \Rightarrow y-8 = -2\sqrt{x^2+(y-2)^2}$$

$$\Rightarrow y^2-16y+64 = 4\left(x^2+y^2-4y+4\right) \qquad \Rightarrow 4x^2+3y^2 = 48$$

$$\Rightarrow \frac{x^2}{12}+\frac{y^2}{16}=1$$

Further investigation

Do you think that $|z+2i|+|z-2i| = 2$ can give a hyperbola : $y^2-\frac{x^2}{3}=1$?